Exercises for Chapter 2, *Applied Control Theory for Embedded Systems*

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An addendum to Applied Control Theory for Embedded Systems.

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Chapter 2 Problems

1. Derive the z transforms in table 2.1 on pp 23 and 24, equations 2.38, 2.42, 2.44 and 2.45.

2. For each of the following z-domain signals, expand the signal into the form

$$H(z) = A_0 + \frac{A_1 z}{z - a_1} + \frac{A_2 z}{z - a_2} + \cdots \quad (1)$$

   using the method specified in each case.

   (a) Using the method shown in Example 2.2 on page 20:

$$H(z) = \frac{z}{z^2 - 1.9z + 0.9754} \quad (2)$$

   (b) Using the method[1] presented on pp 21-23:

$$H(z) = \frac{0.2z^2 + 0.2z}{z^3 - 2.6z^2 + 2.24z - 0.64} \quad (3)$$

   (c) Using Scilab's `pfss` method[2,3]

$$H(z) = \frac{z^3}{z^3 - 2.4z^2 + 1.91z - 0.504} \quad (4)$$

3. For each of the three signals above, find the signal's partial fraction expansion. Use the z transform pairs[4] in Table 2.1 on pp 23-24.

4. Final Value Theorem

   (a) Use the final value theorem to find the value at which the time-domain signal settles out in 2a, 2b and 2c. Use the equations as shown, not the answers from problem 2.

   (b) Check your answers against the time domain expressions in 3.

[1]Note that there are some errors in the discussion in the book. See [errata]. In particular, note that (2.27) should read

$$A_1 = \lim_{z \to a_1} \left[\frac{z - a_1}{z} X(z) \right]$$

[2]http://www.scilab.org

[3]Note that this will take some messing around with `pfss`, both to coerce the formula into the correct form, and because you will have to set the `rmax` parameter in the function call quite high to get good results.

[4]Note that (2.43) is in error, and should read:

$$\mathcal{Z}\left\{k\, d^k\, u(k)\right\} = \frac{dz}{(z - d)^2}$$

Refer to [errata] for details.

5. Initial Value Theorem

   (a) Use the initial value theorem to find the value of the time-domain signal at k = 0 in 2a, 2b and 2c. Use the equations as shown, not the answers from problem 2.

   (b) Check your answer against the time domain expressions in 3.

6. Consider the transfer function

$$H(z) = \frac{0.2z^2 + 0.2z}{z^3 - 2.6z^2 + 2.24z - 0.64} \tag{5}$$

   write a difference equation that describes the system in the time domain.

7. Consider the following difference equation describing the behavior of a system:

$$y_k = [1.8 + 0.15\,\cos(0.1\,k)]\,y_{k-1} - 0.9\,y_{k-2} + 0.95\left(x_k^2 + 1\right) \tag{6}$$

   (a) Can this difference equation be made into a transfer function as-is? Why or why not?

   (b) If we ignore the effect of the time-varying parameter and we linearize around x = 1, (6) becomes

$$y_k = 1.8\,y_{k-1} - 0.9\,y_{k-2} + 1.9\,x_k \tag{7}$$

   Write the transfer function of this linearized, time invariant system.

   (c) Explain how one might go about getting from (6) to (7).

   (d) Under what circumstances may the approximation in (7) be valid? When may they not be valid?

8. Stability

   (a) Is the difference equation you wrote in problem 6 stable? Why or why not?

   (b) Is the system from problem 7b stable? Why or why not?

   (c) In problem 7b we make the assumption that the value of the cosine term averages out to zero, justifying the factor of $1.8\,y_{k-1}$ in (7). But this is only the case if the system so described varies slowly with respect to the frequency of the sinusoidal modulation. A more pessimistic (and thus potentially safer) value to assume is $1.95\,y_{k-1}$. With this substitution made, 7 becomes

$$y_k = 1.95\,y_{k-1} - 0.9\,y_{k-2} + 1.9\,x_k \tag{8}$$

   Is the system, as described, stable?

   (d) Which part of a system's transfer function determines it's stability properties?

   (e) What sort of input signal will cause a stable, linear, time-invariant system to become unstable?

(f) What is the critical value of a system pole that determines whether the pole is stable?

(g) If a nonlinear system is stable and well behaved for a given input signal, does that mean it will be stable for all input signals?

9. Consider a pair of systems with transfer functions

$$H_1(z) = \frac{0.1465z}{z^2 - 1.74z + 0.8865} \tag{9}$$

and

$$H_2(z) = \frac{0.0025z}{z^2 - 1.884z + 0.8865} \tag{10}$$

(a) Find the response of each system for the frequencies $\theta \in \{0, \pi/8, \pi/4, 3\pi/8, \pi/2, 3\pi/4, 7\pi/8, 5\pi/4, -\pi/8\}$ (hint: Scilab will be a big help, here).

(b) Use SciLab to generate Bode plots for the two systems. Compare and contrast the two responses.

(c) Use SciLab to generate the unit step response for the two systems. Compare and contrast the two responses.

Chapter 2 Answers

Problem 1:

These transforms can all be derived using (2.18) on page 18, which is the z transform for

$$x_k = \begin{cases} 0 & k < 0 \\ a^k & k \geq 0 \end{cases} \tag{11}$$

That discussion is a bit terse, so I'll expand it here: The z transform for (11) is arrived at by plugging in the expression for x_k into (2.16) to get

$$X(z) = \mathcal{Z}\{z_k\} = \sum_{k=0}^{\infty} a^k z^{-k} \tag{12}$$

Since Wikipedia has destroyed all initiative by putting all human knowledge[5] on the web, I'll just cite their page on the geometric series, and show their result ([WikiGeo]):

$$\sum_{k=0}^{\infty} ar^k = \frac{a}{1-r} \tag{13}$$

If one is careful about the variable collision between (12) and (13), and lets the a in (13) be equal to 1, and the r in (13) be equal to a/z from (12), then we get

$$X(z) = \mathcal{Z}\{z_k\} = \sum_{k=0}^{\infty} a^k z^{-k} = \frac{1}{1-\frac{a}{z}} \tag{14}$$

Multiply this through by z in the numerator and denominator, and you get $X(z) = \frac{z}{z-a}$, which is just (2.18).

Equation (2.38)

To find the z transform for the unit step function in (2.38), I simply note that it matches my signal x_k above with $a = 1$:

$$\mathcal{Z}\{u(k)\} = \sum_{k=0}^{\infty} 1^k z^{-k} = \frac{z}{z-1}$$

[5] Or at least that version of it that has been written down by the latest author...

Equation (2.42)

To find the z transform of d_k in (2.42) I simply replace the a in (2.18) with d:

$$\mathcal{Z}\left\{d^k u\left(k\right)\right\} = \sum_{k=0}^{\infty} d^k z^{-k} = \frac{z}{z-d}$$

Equation (2.44)

To solve (2.44) we must first use Euler's formula on $\cos\left(\theta\, k\right)$ to put it into terms of the complex exponential:

$$\cos\left(\theta\, k\right) = \frac{1}{2}\left(e^{j\theta k} + e^{-j\theta k}\right)$$

From this it follows that

$$d^k \cos\left(\theta\, k\right) = \frac{d^k}{2}\left(e^{j\theta k} + e^{-j\theta k}\right) = \frac{\left(d\, e^{j\theta}\right)^k}{2} + \frac{\left(d\, e^{-j\theta}\right)^k}{2} \tag{15}$$

Now apply (2.18) to both terms on the right of (15) to get

$$\mathcal{Z}\left\{d^k \cos\left(\theta\, k\right)\right\} = \mathcal{Z}\left\{\frac{\left(d\, e^{j\theta}\right)^k}{2} + \frac{\left(d\, e^{-j\theta}\right)^k}{2}\right\} = \frac{1}{2}\frac{z}{z - d\, e^{j\theta}} + \frac{1}{2}\frac{z}{z - d\, e^{-j\theta}}$$

Now it's just arithmetic:

$$\frac{1}{2}\frac{z}{z - d\, e^{j\theta}} + \frac{1}{2}\frac{z}{z - d\, e^{-j\theta}} = \frac{1}{2}\frac{z\left(z - d\, e^{-j\theta}\right) + z\left(z - d\, e^{j\theta}\right)}{\left(z - d\, e^{j\theta}\right)\left(z - d\, e^{-j\theta}\right)} = \frac{1}{2}\frac{z^2 + z^2 - d\left(e^{j\theta} + e^{-j\theta}\right)z}{z^2 - d\left(e^{j\theta} + e^{-j\theta}\right)z + d^2 e^{j\theta} e^{-j\theta}}$$

Note that both the numerator and the have elements that can be reduced using Euler's formula, and that the z^0 term in the denominator has exponential terms that cancel. When we carry out all the simplifications we are left with:

$$\mathcal{Z}\left\{d^k \cos\left(\theta\, k\right)\right\} = \frac{1}{2}\frac{z^2 + z^2 - d\left(e^{j\theta} + e^{-j\theta}\right)z}{z^2 - d\left(e^{j\theta} + e^{-j\theta}\right)z + d^2 e^{j\theta} e^{-j\theta}} = \frac{z^2 - \left(d\, \cos\theta\right)z}{z^2 - 2\left(d\, \cos\theta\right)z + d^2}$$

Which is what we set out to show.

Equation (2.45)

This is similar to the solution of (2.44), except that we start with Euler's formula for $\sin \theta k$:

$$\sin(\theta k) = \frac{e^{j\theta k} - e^{-j\theta k}}{2j}$$

This leads to

$$d^k \sin(\theta k) = d^k \frac{e^{j\theta k} - e^{-j\theta k}}{2j} = \frac{\left(de^{j\theta}\right)^k}{2j} - \frac{\left(de^{-j\theta}\right)^k}{2j}$$

We can apply the fact that the z transform is linear, plus equation (2.18), to find that

$$\mathcal{Z}\left\{d^k \sin(\theta k)\right\} = \mathcal{Z}\left\{\frac{\left(de^{j\theta}\right)^k}{2j} - \frac{\left(de^{-j\theta}\right)^k}{2j}\right\} = \frac{1}{2j}\frac{z}{z - de^{j\theta}} - \frac{1}{2j}\frac{z}{z - de^{-j\theta}}$$

then we can employ arithmetic skills to find

$$\frac{1}{2j}\frac{z}{z - de^{j\theta}} - \frac{1}{2j}\frac{z}{z - de^{-j\theta}} = \frac{1}{2j}\frac{z\left(z - de^{-j\theta}\right) - z\left(z - de^{j\theta}\right)}{\left(z - de^{j\theta}\right)\left(z - de^{-j\theta}\right)} = \frac{1}{2j}\frac{z^2 - z^2 + d\left(e^{j\theta} - e^{-j\theta}\right)z}{z^2 - d\left(e^{j\theta} + e^{-j\theta}\right)z + d^2 e^{j\theta} e^{-j\theta}}$$

Note that both the numerator and the have elements that can be reduced using Euler's formula, and that the z_0 term in the denominator has exponential terms that cancel. When we carry out all the simplifications we are left with:

$$\mathcal{Z}\left\{d^k \sin(\theta\, k)\right\} = \frac{1}{2j}\frac{z^2 - z^2 + d\left(e^{j\theta} + e^{-j\theta}\right)z}{z^2 - d\left(e^{j\theta} + e^{-j\theta}\right)z + d^2 e^{j\theta} e^{-j\theta}} = \frac{d\, \sin(\theta)\, z}{z^2 - 2\left(d\, \cos\theta\right)z + d^2}$$

which is, once again, what we set out to show.

Problem 2a

Recall that (2) states that $H(z) = \frac{z}{z^2 - 1.9z + 0.9754}$. The roots[6] of the system polynomial of $H(z)$ are $z \in \{0.95 \pm j0.27\}$, so

$$H(z) = \frac{z}{(z - 0.95 - j0.27)(z - 0.95 + j0.27)}$$

Now I put this in the form of (2.22):

$$H(z) = \frac{z}{(z - 0.95 - j0.27)(z - 0.95 + j0.27)} = \frac{A_1 z}{z - 0.95 - j0.27} + \frac{A_2 z}{z - 0.95 + j0.27}$$

[6] Found using the Scilab command

```
roots(%z^2 - 1.9*%z + 0.9754)
```

I multiply out the right hand side to get

$$H(z) = \frac{(A_1 + A_2) z^2 + [(-0.95 + j0.27) A_1 + (-0.95 - j0.27) A_2] z}{z^2 - 1.9z + 0.9754} \tag{16}$$

After collecting terms in the numerator and equating (2) with (16) I get

$$\frac{z}{z^2 - 1.9z + 0.9754} = \frac{(A_1 + A_2) z^2 + [(-0.95 + j0.27) A_1 + (-0.95 - j0.27) A_2] z}{z^2 - 1.9z + 0.9754}$$

Any time you are faced with equating two polynomials, you must equate like exponents of the variable. In this case we can apply the rule to get equations for A_1 and A_2:

$$(A_1 + A_2) z^2 = 0$$
$$[(-0.95 + j0.27) A_1 + (-0.95 - j0.27) A_2] z = z$$

Now we can strip out the dependence on z to get a system of equations:

$$A_1 + A_2 = 0$$
$$(-0.95 + j0.27) A_1 + (-0.95 - j0.27) A_2 = 1$$

We can either solve this using matrix operations in a suitable math package, or we can note that the upper equation above assures that $A_1 = -A_2$; substituting in the value for A_2 into the lower equation gets us

$$(-0.95 + j0.27) A_1 + (-0.95 - j0.27)(-A_1) = 1 = -0.95A_1 + 0.95A_1 + j0.27A_1 + j0.27A_1$$

Solving, we find that $j0.54\, A_1 = 1$, or $A_1 = -j1.852$. From this we find

$$H(z) = \frac{j1.852\, z}{z - 0.95 + j0.27} - \frac{j1.852\, z}{z - 0.95 - j0.27} \tag{17}$$

Problem 2b

Recall that $H(z) = \frac{0.2z^2 + 0.2z}{z^3 - 2.6z^2 + 2.24z - 0.64}$. This transfer function has poles at $z \in \{0.8, 0.8, 1\}$. This gives us

$$H(z) = \frac{0.2z^2 + 0.2z}{(z-1)(z-0.8)^2} = \frac{A_1 z}{z-1} + \frac{A_2 z}{(z-0.8)^2} + \frac{A_3 z}{z - 0.8}$$

We wish to find the residues A_1, A_2, and A_3. The first is easy:

$$A_1 = \lim_{z \to 1} \left[\frac{z-1}{z} \frac{0.2z^2 + 0.2z}{(z-1)(z-0.8)^2} \right]$$

The $z - 1$ terms cancel out, as does one of the z terms in the numerator:

$$A_1 = \lim_{z \to 1} \left[\frac{z-1}{z} \frac{0.2z^2 + 0.2z}{(z-1)(z-0.8)^2} \right] = \lim_{z \to 1} \left(\frac{0.2z + 0.2}{(z-0.8)^2} \right)$$

Taking the limit is now just a matter of substituting 1 for z:

$$A_1 = \frac{0.2\,(1) + 0.2}{(1 - 0.8)^2} = \frac{0.4}{(0.2)^2} = \frac{0.4}{0.04} = 10$$

Finding the A_2 term is also fairly straightforward; following (2.34) we get

$$A_2 = \lim_{z \to 0.8} \left[\frac{(z-0.8)^2}{z} \frac{0.2z^2 + 0.2z}{(z-1)\,(z-0.8)^2} \right]$$

Through an argument very similar to the one which found us A_1 we end up with

$$A_2 = \lim_{z \to 0.8} \left[\frac{(z-0.8)^2}{z} \frac{0.2z^2 + 0.2z}{(z-1)\,(z-0.8)^2} \right] = \lim_{z \to 0.8} \frac{0.2z + 0.2}{z - 1}$$

or

$$A_2 = \left. \frac{0.2z + 0.2}{z - 1} \right|_{z=0.8} = \frac{0.2\,(0.8) + 0.2}{0.8 - 1} = -1.8$$

Finding A_3 is a bit difficult, but difficulty in pursuit of lofty goals builds character. So we'll proceed. Using (2.35) we find that

$$A_3 = \lim_{z \to 0.8} \left[\frac{d}{dz} \frac{(z-0.8)^2}{z} \frac{0.2z^2 + 0.2z}{(z-1)\,(z-0.8)^2} \right]$$

Some of this work is already done for us, in the process of finding A_2:

$$A_3 = \lim_{z \to 0.8} \left[\frac{d}{dz} \frac{(z-0.8)^2}{z} \frac{0.2z^2 + 0.2z}{(z-1)\,(z-0.8)^2} \right] = \lim_{z \to 0.8} \left(\frac{d}{dz} \frac{0.2z + 0.2}{z - 1} \right) \tag{18}$$

It's still up to us to find the derivative, however:

$$\frac{d}{dz} \frac{0.2z + 0.2}{z - 1} = \frac{0.2\,(z-1) - (0.2z + 0.2)}{(z-1)^2} = \frac{0.2z - 0.2z - 0.2 - 0.2}{(z-1)^2} = -\frac{0.4}{(z-1)^2}$$

Substituting this back into (18) we get

$$A_3 = \lim_{z \to 0.8} \left(\frac{d}{dz} \frac{0.2z + 0.2}{z - 1} \right) = -\left. \frac{0.4}{(z-1)^2} \right|_{z=0.8} = -\frac{0.4}{(0.2)^2} = -10$$

Wow! We have all of them! Putting this together we get

$$H\,(z) = \frac{0.2z^2 + 0.2z}{(z-1)\,(z-0.8)^2} = \frac{10z}{z-1} - \frac{1.8z}{(z-0.8)^2} - \frac{10z}{z-0.8} \tag{19}$$

Problem 2c

Recall that according to (4), $H(z) = \frac{z^3}{z^3 - 2.4z^2 + 1.91z - 0.504}$.

With Scilab, the details aren't nearly so onerous, but we miss out on the beauty of the mathematics as we go by. It is also necessary to remember that the Scilab function that finds partial fraction expansions, `pfss`, has some quirks that we must keep in mind. The first is that it 'thinks' that we're only interested in continuous-time systems, so it will insist on translating the polynomials into s, even if we present the problem with polynomials in z. The second is that it wants to do partial fraction expansion the way mathematicians usually do: as a series ${}^{A_1}/_{s+a_1} + {}^{A_2}/_{s+a_2} + \cdots$. But we want to have that z in our numerator, for our convenience. The third problem is that by default, `pfss` does not play well with the transfer functions we give it.

The first fault can be dealt with by holding your mouth right: just look at the polynomials in s and take them to be polynomials in z. The second problem can be dealt with by dividing the signal by z in the function call, and multiplying it back in later as you write down the results. The last is dealt with by calling `pfss` with a second, optional, parameter that tells it to look closer. This optional parameter is called `rmax`, and I have found that a value of 10^6 works well.

So, put these together, and we get the following Scilab session:

```
-->H = %z^3 / (%z^3 - 2.4 * %z^2 + 1.91 * %z - 0.504)
 H  =
 
               3
              z
    -----------------------
                        2    3
  - 0.504 + 1.91z - 2.4z + z

-->pfss(H / %z, 1e6)
 ans  =


       ans(1)

     40.5
    ------
  - 0.9 + s

       ans(2)

    - 64
    ------
  - 0.8 + s

       ans(3)

     24.5
    ------
  - 0.7 + s
 -->
```

Changing the s back to z and multiplying through by z, we get

$$H(z) = \frac{40.5z}{z-0.9} - \frac{64z}{z-0.8} + \frac{24.5z}{z-0.7} \tag{20}$$

Problem 3

Problem 3a

The *hard* way to do this is to use (17), apply (2.42) twice, then do a bunch of arithmetic on complex numbers to get an answer. The *easy* way to do this is to note that the form of (2) is very similar to that of (2.45). So one must just equate them:

$$H(z) = \frac{z}{z^2 - 1.9z + 0.9754} = \frac{A\, d\, \sin(\theta)\, z}{z^2 - 2\, d\, \cos(\theta)\, z + d^2} \tag{21}$$

From this we can compare like portions of the numerator and denominator to extract a system of equations:

$$d^2 = 0.9754$$
$$2\,d\,\cos\theta = 1.9$$

Solving, we get $d = \sqrt{0.9754} =$ and $\theta = \cos^{-1}\frac{1.9}{2\sqrt{0.9754}}$. Then from the numerator we can extract the gain:

$$A\,\sqrt{0.9754}\sqrt{1-\left(\frac{1.9}{2\sqrt{0.9754}}\right)^2} = 1$$

The left hand simplifies:

$$A\,\sqrt{0.9754}\sqrt{1-\left(\frac{1.9}{2\sqrt{0.9754}}\right)^2} = A\sqrt{0.9754-\left(\frac{1.9}{2}\right)^2} = 0.27\,A$$

so $A = {}^{1}/_{0.27} = 3.704$. Applying (2.45) in reverse, we get

$$x_k = 0.9754^{k/2}\,\sin\left(\theta\right)\,u\left(k\right) \tag{22}$$

with $\theta = \cos^{-1}\frac{1.9}{2\sqrt{0.9754}}$, or approximately 0.2769 radians.

Problem 3b

Recall that (19) says that $H\left(z\right) = \frac{10z}{z-1} - \frac{1.8z}{\left(z-0.8\right)^2} - \frac{10z}{z-0.8}$. In this case the easy answer is to start with the expanded form. Matching the three terms on the right side of (19) to Table 2.1 on pp 23 and 24, we see that the first and last terms are in the correct form, but the middle one must be massaged. We need to equate

$$-\frac{1.8z}{\left(z-0.8\right)^2} = \frac{A\,d\,z}{\left(z-d\right)^2}$$

Clearly $d = 0.8$. Then $A\ \left(0.8\right) = -1.8$, and $A = -{}^{1.8}/_{0.8} = -2.25$. Then

$$H\left(z\right) = \frac{10z}{z-1} - \frac{2.25\ \left(0.8\right)\ z}{\left(z-0.8\right)^2} - \frac{10z}{z-0.8}$$

From this it follows that the time-domain signal is

$$x_k = \left(10 - 2.25\ \left(0.8\right)^k\ k - 10\ \left(0.8\right)^k\right)u\left(k\right) \tag{23}$$

Problem 3c

Recall that (20) is $H\left(z\right) = \frac{40.5z}{z-0.9} - \frac{64z}{z-0.8} + \frac{24.5z}{z-0.7}$. We can almost read this off directly, and use (2.42) three times to get

$$x_k = \left[40.5\left(0.9\right)^k - 64\left(0.8\right)^k + 24.5\left(0.7\right)^k\right]u\left(k\right) \tag{24}$$

Problem 4

From Problem 2a

Recall that (2) states that $H(z) = \frac{z}{z^2-1.9z+0.9754}$. Applying the final value theorem (p 28), we get

$$\lim_{k\to\infty} h_k = \lim_{z\to 1}\left(\frac{z-1}{z}H(z)\right) = \lim_{z\to 1}\left(\frac{z-1}{z}\frac{z}{z^2-1.9z+0.9754}\right)$$

A quick check shows that the denominator of the fraction doesn't go to zero as z goes to 1, so the math is simple:

$$\lim_{k\to\infty} h_k = \left.\frac{z-1}{z}\frac{z}{z^2-1.9z+0.9754}\right|_{z=1} = \frac{(0)(1)}{(1)(1-1.9+0.9754)} = \frac{0}{0.08} = 0$$

This jibes with the time-domain answer, which had a decaying exponential but no constant term.

From Problem 2b

Recall that $H(z) = \frac{0.2z^2+0.2z}{z^3-2.6z^2+2.24z-0.64}$. Applying the final value theorem (p 28), we get

$$\lim_{k\to\infty} h_k = \lim_{z\to 1}\left(\frac{z-1}{z}H(z)\right) = \lim_{z\to 1}\left(\frac{z-1}{z}\frac{0.2z^2+0.2z}{z^3-2.6z^2+2.24z-0.64}\right)$$

In this case, we note that the denominator of our signal's z transform *does* go to zero, so we must actually compute the limit. We can go two ways with this. One is to observe that because the denominator goes to zero as z goes to 1 it must have at least one root at $z = 1$. Then we can try dividing the root out. We do so, and get

$$\frac{z-1}{z}\frac{0.2z^2+0.2z}{z^3-2.6z^2+2.24z-0.64} = \frac{1}{z}\frac{0.2z^2+0.2z}{z^2-1.6z+0.64}$$

To our delight, we note that the denominator of *this* expression is nonzero, which means that the limit is easy:

$$\lim_{k\to\infty} h_k = \left.\frac{1}{z}\frac{0.2z^2+0.2z}{z^2-1.6z+0.64}\right|_{z=1} = \frac{(1)(0.2+0.2)}{1-1.6+0.64} = \frac{0.4}{0.04} = 10$$

This jibes with the time-domain answer, which had a decaying exponential term which goes to zero, k times a decaying exponential which also goes to zero, and a constant term of 10.

An alternative to the above polynomial division in the denominator is to use l'Hosptal's rule to evaluate the limit:

$$\lim_{z\to 1}\left(\frac{z-1}{z}\frac{0.2z^2+0.2z}{z^3-2.6z^2+2.24z-0.64}\right) = \lim_{z\to 1}\left(\frac{\frac{d}{dz}(z-1)(0.2z^2+0.2z)}{\frac{d}{dz}z(z^3-2.6z^2+2.24z-0.64)}\right)$$

Doing the inner calculation we get

$$\frac{\frac{d}{dz}\left(z-1\right)\left(0.2z^2+0.2z\right)}{\frac{d}{dz}z\left(z^3-2.6z^2+2.24z-0.64\right)}=\frac{\left(0.2z^2+0.2z\right)+\left(z-1\right)\left(0.4z+0.2\right)}{\left(z^3-2.6z^2+2.24z-0.64\right)+z\left(3z^2-5.2z+2.24\right)}$$

Now cross our fingers and hope it was only one root at z = 1, and calculate:

$$\lim_{k\to\infty}h_k=\left.\frac{\left(0.2z^2+0.2z\right)+\left(z-1\right)\left(0.4z+0.2\right)}{\left(z^3-2.6z^2+2.24z-0.64\right)+z\left(3z^2-5.2z+2.24\right)}\right|_{z=1}$$

We find no unsightly zeros in the denominator, so we can evaluate

$$\lim_{k\to\infty}h_k=\frac{0.2+0.2+\left(1-1\right)\left(0.4+0.2\right)}{\left(1-2.6+2.24-0.64\right)+\left(1\right)\left(3-5.2+2.24\right)}=\frac{0.4}{0.04}=10$$

This agrees — with much more work — with the results we got from factoring the denominator.

From Problem 2c

Recall that according to (4), $H\left(z\right)=\frac{z^3}{z^3-2.4z^2+1.91z-0.504}$. Applying the final value theorem (p 28), we get

$$\lim_{k\to\infty}h_k=\lim_{z\to 1}\left(\frac{z-1}{z}H\left(z\right)\right)=\lim_{z\to 1}\left(\frac{z-1}{z}\frac{z^3}{z^3-2.4z^2+1.91z-0.504}\right)$$

A quick check shows that the denominator of the fraction doesn't go to zero as z goes to 1, so the math is simple:

$$\lim_{k\to\infty}h_k=\left.\frac{z-1}{z}\frac{z^3}{z^3-2.4z^2+1.91z-0.504}\right|_{z=1}=\frac{\left(0\right)\left(1\right)}{\left(1\right)\left(1-2.4+1.91-0.504\right)}=\frac{0}{0.006}=0$$

This jibes with the time-domain answer, which had three decaying exponential terms but no constant term.

Problem 5a

From Problem 2a

Recall that (2) states that $H\left(z\right)=\frac{z}{z^2-1.9z+0.9754}$.

$$h_0=\lim_{z\to\infty}H\left(z\right)=\lim_{z\to 1}\left(\frac{z}{z^2-1.9z+0.9754}\right)$$

Per the discussion right after (2.64) on page 29, the leading term of the numerator is in z while the leading term in the denominator is in z^2, so the initial value must be zero. This jibes with the time-domain answer, which, being multiplied by $sin(\theta\,k)$, must be zero at $k=0$.

From Problem 2b

Recall that $H(z) = \frac{0.2z^2+0.2z}{z^3-2.6z^2+2.24z-0.64}$.

$$h_0 = \lim_{z\to\infty} H(z) = \lim_{z\to 1}\left(\frac{0.2z^2+0.2z}{z^3-2.6z^2+2.24z-0.64}\right)$$

Per the discussion right after (2.64) on page 29, the leading term of the numerator is in z^2 while the leading term in the denominator is in z^3, so the initial value must be zero. This jibes with the time-domain answer, which at $k = 0$ is $10–1.8(0)(0.8)^0–10(0.8)^0 = 10–0–10 = 0$.

From Problem 2c

Recall that according to (4), $H(z) = \frac{z^3}{z^3-2.4z^2+1.91z-0.504}$.

$$h_0 = \lim_{z\to\infty} H(z) = \lim_{z\to 1}\left(\frac{0.2z^2+0.2z}{z^3-2.6z^2+2.24z-0.64}\right)$$

Per the discussion right after (2.64) on page 29, the leading term of the numerator is z^3, and so is the leading term in the denominator. Thus, the initial value must be one. This jibes with the time-domain answer, which at $k = 0$ is $40.5 - 64 + 24.5 = 1$.

Problem 6

Recall that (5) states that $H(z) = \frac{0.2z^2+0.2z}{z^3-2.6z^2+2.24z-0.64}$.

To make a difference equation from this, we must first make it into an equation in z that specifies input and output signals. Let

$$H(z) = \frac{X(z)}{U(z)} = \frac{0.2z^2+0.2z}{z^3-2.6z^2+2.24z-0.64}$$

Now rearrange terms by cross-multiplying, to get

$$\left(z^3-2.6z^2+2.24z-0.64\right)X(z) = \left(0.2z^2+0.2z\right)U(z)$$

and separate each term

$$z^3X(z) - 2.6\,z^2X(z) + 2.24\,z\,X(z) - 0.64\,X(z) = 0.2\,z^2U(z) + 0.2\,z\,U(z)$$

Divide through by z^3:

$$X(z) - 2.6\,z^{-1}X(z) + 2.24\,z^{-2}X(z) - 0.64\,z^{-3}X(z) = 0.2\,z^{-1}U(z) + 0.2\,z^{-2}U(z)$$

Now do the inverse z transform by inspection, using (2.69):

$$x_k - 2.6\, x_{k-1} + 2.24\, x_{k-2} - 0.64\, x_{k-3} = 0.2\, u_{k-1} + 0.2\, u_{k-2}$$

To finish, collect all of the terms except for the present value of the output on the right:

$$x_k = 2.6\, x_{k-1} - 2.24\, x_{k-2} + 0.64\, x_{k-3} + 0.2\, u_{k-1} + 0.2\, u_{k-2}$$

Done. Note that as a practical matter (see Chapter 10) actually realizing this filter in this form would be unwise — the correct way would be to factor it into three 1^{st}-order filters, and cascade them.

Problem 7

Recall that (6) reads $y_k = [1.8 + 0.15\, \cos(0.1\, k)]\, y_{k-1} - 0.9\, y_{k-2} + 0.95\, (x_k^2 + 1)$.

Problem 7a

This difference equation cannot be made into a transfer function both because it is nonlinear, and because it is time varying.

Problem 7b

Recall that (7) reads $y_k = 1.8\, y_{k-1} - 0.9\, y_{k-2} + 1.9\, x_k$. Take the z transform element-by-element to get

$$Y(z) = 1.8\, z^{-1} Y(z) - 0.9\, z^{-2} Y(z) + 1.9\, X(z)$$

rearrange, to bring like terms on like sides

$$\left(1 - 1.8\, z^{-1} + 0.9\, z^{-2}\right) Y(z) = 1.9\, X(z)$$

now divide by $X(z)$ and by the term on the left

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1.9}{1 - 1.8\, z^{-1} + 0.9\, z^{-2}}$$

In this form things would look nice to a DSP expert[7], but for us control guys let's multiply top and bottom by z^2 to get

$$H(z) = \frac{1.9\, z^2}{z^2 - 1.8\, z + 0.9} \tag{25}$$

[7] DSP folks often seem to like to see polynomials in terms of $1/z$.

Problem 7c

In general, to go from a nonlinear differential equation to a linear one, you find the derivative with respect to the variables in question around some operating point. In this case the only nonlinearity is in x, and the only term in x is the one that reads $0.95\left(x^2+1\right)$. So to approximate this we note that the slope of this term is 1.9, and that when $x = 1$, $1.9x = 0.95\left(x+1\right)$. Thus, for values of x close to 1

$$y_k \simeq \left[1.8 + 0.15\,\cos\left(0.1\,k\right)\right] y_{k-1} - 0.9\,y_{k-2} + 1.9x_k$$

This approximation is linear, but it is still time varying. To eliminate the time varying part, we take a leap, and claim that the time variation due to the cosine term is insignificant, and we set that term to zero, giving us

$$y_k \simeq 1.8y_{k-1} - 0.9\,y_{k-2} + 1.9x_k$$

Problem 7d

If the variation of the cosine term in (6) is faster than the other dynamics in the system then its action can be considered to average out, and the analysis can often be carried out assuming no time variance. Alternately, if the system is embedded in a control loop that controls its behavior in a manner that is relatively insensitive to the value of the cosine term, then that term may be safely ignored.

If the value of the x^2 term doesn't change much during the operation of the system — meaning that x stays close to 1, then the nonlinearity won't make much difference. Note the lack of engineering terms here — just how much x can deviate from 1 and still maintain proper system behavior depends to a considerable degree on the rest of the system (these issues will be explored in more depth in Chapter 8).

Problem 8

Problem 8a

Recall that the difference equation was written from (6), the denominator of which is $z^3 - 2.6z^2 + 2.24z - 0.64$. This denominator has roots at $x = 1$ and a double root at $x = 0.8$. Because of the root at $x = 1$ the system is not stable[8].

Problem 8b

The denominator of (25) is $z^2 - 1.8\,z + 0.9$, which has roots at $x = 0.9 \pm j0.3$. These roots have an absolute value less than 1, so the system is stable.

[8] Technically it is *metastable*, because the root in question lies on the stability boundary.

Problem 8c

If the y_{k-1} coefficient is taken to be 1.95, then the transfer function becomes $H(z) = \frac{1.9\,z^2}{z^2 - 1.95\,z + 0.9}$, and the system characteristic polynomial becomes $z^2 - 1.95\,z + 0.9$, with roots at $x = 0.75$ and $x = 1.2$. This is an unstable system.

Problem 8d

The system characteristic polynomial, i.e. the transfer function denominator, determins whether the system is stable or not.

Problem 8e

Stability is a global property of linear systems. There is no input signal that will cause a stable linear system to become unstable.

Problem 8f

A system pole is considered stable if it is inside the stability boundary on the complex plane. The stability boundary is the unit circle, i.e. the locus of all points $|x| = 1$. A system pole is stable if its absolute value is less than one, metastable if its absolute value is equal to one, and unstable if its absolute value exceeds one.

Problem 8g

One of the salient properties of linear systems is that their stability, or lack thereof, is global: a linear system is either stable or not. A nonlinear system, on the other hand, can have stable operating regions and unstable operating regions. Just because a nonlinear system has shown stable behavior in response to a given input signal, does not mean that some other input signal may throw the system into an unstable operating region.

Problem 9

We are finding the frequency and time responses for two systems. Recall that (9) reads $H_1(z) = \frac{0.1465z}{z^2 - 1.74z + 0.8865}$, and (10) reads $H_2(z) = \frac{0.0025z}{z^2 - 1.884z + 0.8865}$.

Problem 9a

Finding a system's response can be done by hand, but it is tedious. I will demonstrated it for these two systems for the frequency $\theta = \pi/2$. At this frequency, we find that $z = e^{j\,\pi/2} = j$.

Applying this to (9) gives us

$$H_1(j) = \frac{j0.1465}{j^2 - j1.74 + 0.8865} = \frac{j0.1465}{-j1.74 + 0.8865 - 1} = \frac{j0.1465}{-j1.74 - 0.1135}$$

Solving the fraction, $H_1(j) = -0.0838 - j0.0055 = 0.084\angle - 176°$.

Similarly,

$$H_2(j) = \frac{j0.0025}{j^2 - j1.884 + 0.8865} = \frac{j0.0025}{-j1.884 + 0.8865 - 1} = \frac{j0.0025}{-j1.884 - 0.1135}$$

solving, we get $H_2(j) = -0.00132 - j0.00008 = 0.00132\angle - 177°$.

We could do this by hand for each frequency, but we could also get Scilab to do the heavy lifting for us. To do so, we use Scilab's built-in transfer function notation, and the `horner` function:

```
-->H1 = syslin('d', 0.1465 * %z / (%z^2 - 1.74 * %z + 0.8865));

-->H2 = syslin('d', 0.0025 * %z / (%z^2 - 1.884 * %z + 0.8865));

-->w = %pi * [0 1/8 1/4 3/8 1/2 3/4 7/8 5/4 -1/8];

-->[w'/%pi horner(H1, exp(%i * w'))]
 ans  =

    0         1.
    0.125     0.2241019 - 3.3579333i
    0.25    - 0.3472336 - 0.0686326i
    0.375   - 0.1423895 - 0.0146660i
    0.5     - 0.0838387 - 0.0054688i
    0.75    - 0.0476260 - 0.0012434i
    0.875   - 0.0420561 - 0.0005245i
    1.25    - 0.0476260 + 0.0012434i
  - 0.125     0.2241019 + 3.3579333i

-->[w'/%pi horner(H2, exp(%i * w'))]
 ans  =

    0         1.
    0.125   - 0.0161842 - 0.0049819i
    0.25    - 0.0044504 - 0.0006493i
    0.375   - 0.0021340 - 0.0001926i
    0.5     - 0.0013222 - 0.0000797i
    0.75    - 0.0007764 - 0.0000194i
    0.875   - 0.0006892 - 0.0000083i
    1.25    - 0.0007764 + 0.0000194i
  - 0.125   - 0.0161842 + 0.0049819i
```

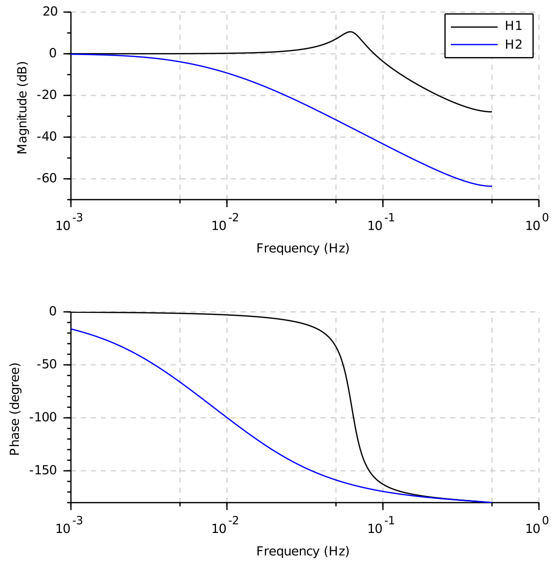

Figure 1: Bode plot of H_1 and H_2.

Problem 9b

Executing the Scilab code

```
-->H1 = syslin('d', 0.1465 * %z / (%z^2 - 1.74 * %z + 0.8865));
-->H2 = syslin('d', 0.0025 * %z / (%z^2 - 1.884 * %z + 0.8865));
-->clf;
-->bode([H1; H2], 0.001, 0.5, 0.01)
-->legend(['H1' 'H2'])
```

will get you the Bode plot shown in Figure 1 on page 19.

Problem 9c

Executing the Scilab code

```
-->k = 0:250;
-->h1 = flts(ones(k), H1); // 'ones(k)' just makes a matrix
-->h2 = flts(ones(k), H2); // of all ones the same size as k
-->clf;
-->plot2d(k, [h1' h2']);
-->legend(['H1' 'H2']);
```

will get you the Bode plot shown in Figure 2 on page 20.

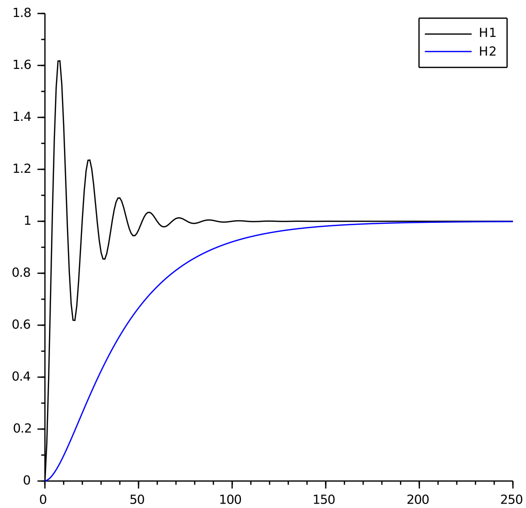

Figure 2: Step responses of H_1 and H_2.

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About the Author

Tim Wescott has 20 years of experience in industry, designing and implementing algorithms and systems for digital signal processing and closed loop servo control. His extensive practical experience in translating concepts from the highly abstract domain of mathematical systems analysis into working hardware gives him a unique ability to make these concepts clear for the working engineer.

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Applied Control Theory for Embedded Systems

This book is written for the practicing engineer who needs to develop working control systems without going back to school for years. It is aimed directly at software engineers who are learning control theory for the first time, however it has found favor engineers of all stripes when they are tasked with implementing control loops. It teaches theory, but it also covers many real-world issues in much greater depth than is taught in the sort of theory course you'd find in many universities.

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